Exact expressions for minor hysteresis loops in the random field Ising model on a Bethe lattice at zero temperature

Prabodh Shukla
Physics Department, North Eastern Hill University, Shillong 793 022, India
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We obtain exact expressions for the minor hysteresis loops in the ferromagnetic random field Ising model on a Bethe lattice at zero temperature in the case when the driving field is cycled infinitely slowly.

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I. INTRODUCTION

Hysteresis is a common phenomenon but uncommonly difficult to treat analytically. Theoretically, hysteresis is expected to vanish as the frequency of the driving field goes to zero, or its period goes to infinity. However, in many cases hysteresis persists over the longest experimental time scales. For example, there are indications that hysteresis would be observed in a permanent magnet even if the applied field were to be cycled so slowly as to take the entire life of an experimentalist to complete one loop. It is of practical importance to make a theory for this kind of hysteresis that persists over the longest practical time scales, and where the effect of temperature on the area and the shape of the hysteresis loop is small. A reasonable starting point for the theory may be the Glauber dynamics of the Ising model at temperature T, driven by a field of frequency ω. However, this is difficult to treat analytically. A simpler version of this model that appears to be adequate for our purpose is obtained by taking the limit $T=0$ first, and then the limit $\omega=0$. The zero-temperature, zero-frequency version produces realistic-looking hysteresis loops if one incorporates a Gaussian distribution of on-site quenched random field in the model. Thus the zero-temperature single-spin-flip dynamics of the ferromagnetic random field Ising model (RFIM) was proposed as a model of hysteresis and Barkhausen noise by Sethna *et al.* [1]. Antiferromagnetic RFIM is also interesting in the context of relaxation without Barkhausen noise [2]. The difficulty in the analytical treatment of the models comes from the presence of the quenched random field. The zero-temperature dynamics of RFIM cannot be solved exactly in two or three dimensions (so far). We have obtained the major hysteresis loop for the ferromagnetic RFIM in one dimension [3] and on a Bethe lattice [4]. The Barkhausen noise on the major loop has been analyzed [5]. Minor hysteresis loops in the ferromagnetic RFIM have been obtained in one dimension [6]. Antiferromagnetic RFIM is apparently more difficult, and its analytic solution is limited so far to the major hysteresis loop in one dimension for a rectangular distribution of the quenched field of width 2Δ where $\Delta \leqslant |J|$, J being the strength of the nearest neighbor interaction.

In the following, we present the solution of minor loops in the ferromagnetic RFIM on a Bethe lattice. This is an extension of the work presented in Ref. [6], and the completion of a problem that remained open in Ref. [4]; therefore the reader is assumed to be familiar with these references. Several important aspects of the analysis and simulations of the model that are discussed in Refs. [4] and [6] are not repeated here in order to save space, or mentioned only briefly for the sake of completeness. It was shown in one dimension [6] that a reversal of the applied field by an amount $2J$ from anywhere on the major loop brings the system on the opposite half of the major loop. The reversed trajectory merges with the opposite half of the major loop for the portion of the reversed field exceeding $2J$. Thus, the width (along the applied field axis) of a minor loop that touches both halves of the major loop (but does not merge with either of them) is constant and equal to $2J$ irrespective of the position of the minor loop. The shape of the minor loop does depend upon its position inside the major loop. This physically interesting result is now shown to hold on all Bethe lattices irrespective of the coordination number z of the lattice.

II. MAJOR HYSTERESIS LOOP

In this section, we recall the model briefly, and the solution of the major hysteresis loop obtained earlier [4]. The RFIM in an external field h_{ext} is characterized by the Hamiltonian

$$H=-J\sum_{i,j} s_i s_j - \sum_i h_i s_i - h_{ext}\sum_i s_i . \qquad (1)$$

The sum in the first term is restricted to pairs of nearest neighbors on a Bethe lattice of coordination number z. The external field h_{ext} is cycled from $-\infty$ to $+\infty$ and back to $-\infty$. This takes the system around its major hysteresis loop. Spins turn up on the lower half of the loop, and turn down again on the upper half. The applied field changes very slowly. Equivalently, at each value of the external field, the system is allowed adequate time to attain a relaxed state with each spin pointing along the net field at its site. In the relaxed state at $h_{ext}=h$, the probability that an arbitrary site i is up is given by

$$p(h)=\sum_{n=0}^{z} \binom{z}{n} [P^*(h)]^n [1-P^*(h)]^{z-n} p_n(h). \qquad (2)$$

Here $P^*(h)$ is the conditional probability that a nearest neighbor of site i is up before site i is relaxed, and $p_n(h)$ is the probability that the site i with quenched field h_i can turn up in applied field h if n of its nearest neighbors are up. The starting state on the lower half of the major loop has all sites down, and the starting state on the upper half has all sites up. Thus, on the lower half of the major loop, $P^*(h)$ denotes the conditional probability that a nearest neighbor of site *i* turns up before site *i*. On the upper half, $P^*(h)$ denotes the conditional probability that a nearest neighbor of site *i* turns down after site *i*. We distinguish the two situations by putting a subscript *l* for the lower half and *u* for the upper half. This gives

$$P_l^*(h)=\sum_{n=0}^{z-1}\binom{z-1}{n}[P_l^*(h)]^n[1-P_l^*(h)]^{z-1-n}p_n(h) \tag{3}$$

and

$$P_u^*(h)=\sum_{n=0}^{z-1}\binom{z-1}{n}[P_u^*(h)]^n[1-P_u^*(h)]^{z-1-n}p_{n+1}(h). \tag{4}$$

We note that $p_{n+1}(h)=p_n(h+2J)$, and therefore $P_u^*(h)=P_l^*(h+2J)$. Here, $P_u^*(h)$ is the conditional probability that when a site that is up on the upper half of the major loop at field *h*, its nearest neighbor is also up. Similarly, $P_l^*(h+2J)$ is the conditional probability that when a site that is down on the lower half of the major loop at field $h+2J$, its nearest neighbor is up.

III. STARTING A MINOR LOOP

Suppose we are on the lower part of the major loop when the applied field is reversed from *h* to h' ($h'\leqslant h$) to generate the upper half of a minor hysteresis loop. We ask the question, what is the probability that an arbitrary site *i* that was up at *h* turns down at h'? In order to compute this probability correctly, we must take into account the irreversibility of the zero-temperature dynamics. Consider a site *i* that is down on the lower half of the major loop at an applied field $h-\delta h$ but turns up at *h*, where δh is an arbitrarily small field. Once site *i* has turned up, it may not turn down immediately if the field is rolled back to the value $h-\delta h$. The reason is as follows. When site *i* turns up, it increases the net field on each of its nearest neighbors by an amount $2J$. The increased field may cause one or more nearest neighbors of site *i* to turn up. Suppose n_a nearest neighbors of site *i* were already up before site *i* turned up, and n_b nearest neighbors turn up after site *i* turns up. Clearly, n_b must lie in the range $0\leqslant n_b\leqslant z-n_a$. The n_b neighbors increase the local field at site *i* by a finite amount $2n_bJ$. Therefore, an infinitesimal decrease in the applied field will not cause site *i* to turn down unless $n_b=0$. A site *i* with $n_b>0$ will turn down in decreasing applied field only after all of the n_b nearest neighbors have turned down. When the field is reversed to $h'<h$, none of the n_a neighbors (which turned up before site *i* turned up) could possibly turn down before site *i* turns down. This leaves the other n_b neighbors that turned up after site *i*. The n_b neighbors turned up because the field on them increased by an amount $2J$ after site *i* turned up. In decreasing field h', the n_b neighbors will turn down before site *i* turns down. At $h'=h-2J$, all of the n_b neighbors would have turned down with the result that the nearest neighbors of a site *i* that are up at $h'=h-2J$ are precisely those that were up before site *i* flipped up. These neighbors will turn down on the reverse trajectory only after site *i* turns down. Thus, given an up site *i* at $h'=h-2J$ on the upper half of the minor loop, the conditional probability $P_u^*(h-2J)$ that a nearest neighbor of *i* is up is equal to $P_l^*(h)$, where $P_l^*(h)$ is the conditional probability that a nearest neighbor of site *i* is up at field *h* given that site *i* is down on the lower half of the major loop at field *h*. The probability that the site *i* is up at $h-2J$ is given by the equation

$$p(h-2J)=\sum_{n=0}^{z}\binom{z}{n}[P_l^*(h)]^n[1-P_l^*(h)]^{z-n}p_n(h-2J) \tag{5}$$

or, using the identity $P_l^*(h)=P_u^*(h-2J)$,

$$p(h-2J)=\sum_{n=0}^{z}\binom{z}{n}[P_u^*(h-2J)]^n \times[1-P_u^*(h-2J)]^{z-n}p_n(h-2J). \tag{6}$$

It follows from the above equation that the reverse trajectory will meet the upper half of the major loop at $h'=h-2J$ and merge with it for $h'<h-2J$. Thus, the task of computing the minor hysteresis loop is reduced to range $h-2J\leqslant h'\leqslant h$. We return to the question asked at the beginning of this section. What is the probability that a site *i* that is up at *h* turns down at h'? This is given by

$$q'(h')=\sum_{n=0}^{z}\binom{z}{n}[P_l^*(h)]^n[D^*(h')]^{z-n}[p_n(h)-p_n(h')]. \tag{7}$$

Here $D^*(h')$ is the probability that a nearest neighbor of site *i* turns down on the reverse trajectory before site *i*. $D^*(h')$ is determined by the equation

$$D^*(h')=\sum_{n=0}^{z-1}\binom{z-1}{n}[P_l^*(h)]^n[1-P_l^*(h)]^{z-1-n} \times[1-p_{n+1}(h)]+\sum_{n=0}^{z-1}\binom{z-1}{n} \times[P_l^*(h)]^n[D^*(h')]^{z-1-n} \times[p_{n+1}(h)-p_{n+1}(h')]. \tag{8}$$

Given a site *i* that is up at *h*, the first sum above gives the conditional probability that a nearest neighbor of site *i* is down at $h'=h$, i.e., at the very start of the reverse trajectory (and hence remains down for $h'<h$). The second sum takes into account the situation that the nearest neighbor in question is up at *h*, but turns down before site *i* turns down on the return loop. Note that *all* the nearest neighbors of a site *i* that turned up after it turned up on the lower major loop must turn down before site *i* turns down on the upper minor loop.

The magnetization on the reverse trajectory is given by

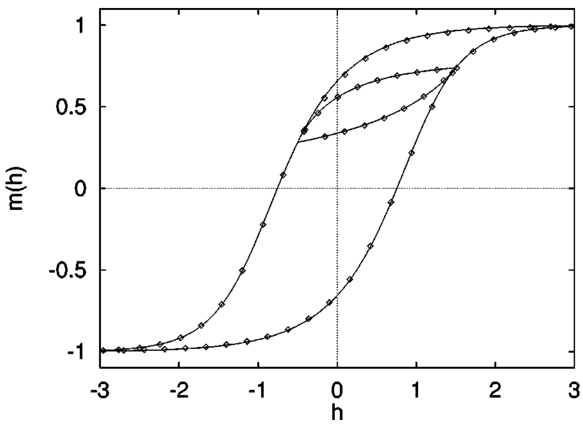

FIG. 1. Major and minor hysteresis loops in RFIM ($J=1$) on a Bethe lattice ($z=3$) for a Gaussian distribution of the random field with mean zero and $\sigma=2$. The minor hysteresis loop is obtained by reversing the applied field from $h=1.5$ to $h'=-0.5$ and back to $h=1.5$. Theoretical result is shown by a continuous line, and symbols show the data obtained from numerical simulation of the model.

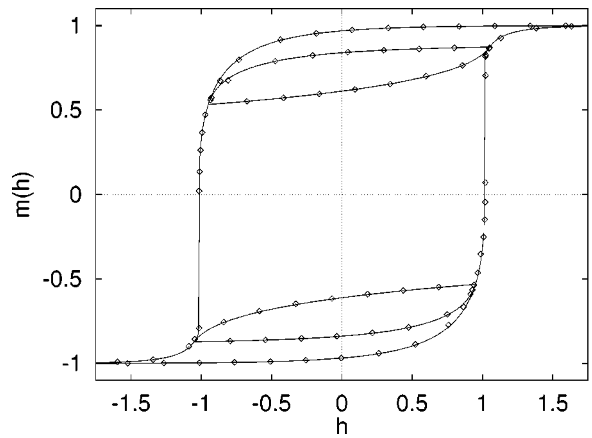

FIG. 2. Hysteresis in RFIM ($J=1$) on a Bethe lattice ($z=4$) for a Gaussian random field of mean zero and $\sigma=1.70$. Discontinuities in the major loop vanish above $\sigma_c=1.78$. Two minor loops are shown starting on the lower major loop at $h=0.95$ and $h=1.05$, respectively. As in Fig. 1, theoretical result is shown by a continuous line, and symbols show the data obtained from numerical simulation of the model. Note that the minor loops touch the upper major loop when the applied field has been reversed by an amount $2J$.

$$m'(h')=2[p(h)-q'(h')]-1. \tag{9}$$

IV. COMPLETING THE MINOR LOOP

We reverse the field h' to h'' ($h''>h'$) to trace the lower half of the return loop. The magnetization on the lower half of the return loop may be written as

$$m''(h'')=2[p(h)-q'(h')+p''(h'')]-1, \tag{10}$$

where $p''(h'')$ is the probability that an arbitrary site i that turned up at h and turned down at h', turns up again at h'',

$$p''(h'')=\sum_{n=0}^{z}\binom{z}{n}[U^*(h'')]^n[D^*(h')]^{z-n}\times[p_n(h'')-p_n(h')]. \tag{11}$$

Here $U^*(h'')$ is the conditional probability that a nearest neighbor of a site i turns up before site i turns up on the lower return loop. It is determined by the equation

$$\begin{aligned}U^*(h'')=P^*(h)&-\sum_{n=0}^{z-1}\binom{z-1}{n}\\&\times[P_l^*(h)]^n[D^*(h')]^{z-1-n}[p_n(h)-p_n(h')]\\&+\sum_{n=0}^{z-1}\binom{z-1}{n}[U^*(h'')]^n[D^*(h')]^{z-1-n}\\&\times[p_n(h'')-p_n(h')].\end{aligned} \tag{12}$$

The rationale behind Eq. (12) is similar to the one behind Eq. (8). Given that a site i is down at h', the first two terms account for the probability that a nearest neighbor of site i is up at $h''\geqslant h'$. Note that the neighbor in question must have been up at h in order to be up at h', and if it is already up at h' then it will remain up on the entire lower half of the return loop, i.e., at $h''\geqslant h'$. The third term gives the probability that the neighboring site was down at h', but turned up on the lower return loop before site i turned up. It can be verified that the lower return loop meets the lower major loop at $h''=h$ and merges with it for $h''>h$, as may be expected on account of the return point memory. The exact expressions given above have been checked against numerical simulations of the model in selected cases resulting in excellent agreement in all cases that were tested.

V. CONCLUDING REMARKS

The method of calculating the minor loop described above may be extended to obtain a series of nested minor loops. The key point is that whenever the applied field is reversed, a site i may flip only after all neighbors of site i that flipped in the wake of site i (on the immediately preceding sector) have flipped back. The neighbors of site i, which remained firm after site i flipped previously, do not yield before site i has flipped again on the return loop. We have obtained the above expression for the return loop when the applied field is reversed from $h_{ext}=h$ on the lower major loop to $h_{ext}=h'$ ($h-2J\leqslant h'\leqslant h$), and reversed again from $h_{ext}=h'$ to $h_{ext}=h''$ ($h''\leqslant h$). When the applied field is reversed a third time from h'' to h''' ($h'''<h''$), expressions for the magnetization on the nested return loop follow the same structure as the one on the trajectory from h to h'. Qualitatively, the role of P^* on the first leg (h to h') is taken up by U^* on the third leg (h'' to h''') of the nested return loop.

We conclude by comparing the results obtained above with numerical simulations on Bethe lattices of coordination number $z=3$ and $z=4$, and also contrast these results with those obtained in the one-dimensional case [6]. Let us specifically choose a Gaussian distribution of the random field with mean value zero and variance σ^2. One generally expects the solution of an Ising model with nearest neighbor interactions on a Bethe lattice ($z \geqslant 3$) to be qualitatively different from its solution in one dimension ($z=2$), and to be similar to the mean-field solution of an infinite-range model. However, these expectations are not borne out in the case of hysteresis in RFIM. The mean-field solution [1] does not show any hysteresis for $\sigma \geqslant \sigma_c(\infty) = \sqrt{2/\pi}$. In contrast to this, there is hysteresis on Bethe lattices for all values of σ. Moreover, the behavior on lattices with $z=2$ and $z=3$ turns out to be qualitatively similar. For lattices with $z \geqslant 4$, the magnetization on each half of the hysteresis loop has a jump discontinuity for $\sigma \leqslant \sigma_c(z)$; the jump discontinuity is absent on lattices with $z \leqslant 3$ for any finite value of σ. Figure 1 shows the major as well as a minor hysteresis loop on a Bethe lattice with $z=3$, and $\sigma=2$. The minor loop starts on the lower half of the major loop at $h=1.5$ and meets the upper half at $h=-0.5$ as may be expected from the theoretical prediction. As the magnetization on the lower half of the major loop is a single valued function of the applied field in Fig. 1, the point where the applied field is reversed determines the minor loop uniquely. The one-dimensional case ($z=2$) is similar [6]. However, the situation is different for $z \geqslant 4$. For $z=4$, we have $\sigma_c(4) = 1.78$ approximately. Figure 2 shows the major loop for $z=4$ and $\sigma=1.70$ with a jump discontinuity at a critical field h_c that is slightly higher than unity for $\sigma=1.70$. There are two values of the magnetization at h_c, both lying on the lower half of the major loop. If we reverse the applied field from the value h_c, we must specify the state of the magnetization from where the return is made, giving us two possible return loops originating at h_c. Figure 2 shows two minor loops starting at $h=0.95$ (slightly less than h_c), and $h=1.05$ (slightly greater than h_c) on the lower half of the major loop. Both the return trajectories touch the upper half of the major loop when the field has been reversed by an amount $2J$ as expected from the theoretical analysis. The overall agreement between the simulations and the theory is also quite good indicating that the model considered here is self-averaging [4–6].

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